ON BANACH SPACES WHOSE DUALS ARE L_1 SPACES

BY

A. J. LAZAR AND J. LINDENSTRAUSS*

ABSTRACT

A structure theorem for Banach spaces whose duals are L_1 spaces, is proved.

The purpose of this note is to settle a question left open in [2, p. 66] as well as a related problem contained implicitly in [3]. A Banach space X is called an $\mathcal{N}_\lambda$ space [2] if there is a net $\{B_\tau\}$ of finite-dimensional subspaces of X directed by inclusion such that $X = \overline{\bigcup_\tau B_\tau}$ and every B_τ is a $\mathcal{P}_\lambda$ space. It was proved in [2, p. 66] that if a Banach space is an $\mathcal{N}_\lambda$ space for every $\lambda > 1$ then X^* is an $L_1(\mu)$ space for some measure μ. Here we shall prove that also the converse is true. In [3] Michael and Pełczyński studied Banach spaces X which have the following property: For every $\varepsilon > 0$ and every finite set A in X there is an integer n and an operator $T: l_n^\infty \to X$ such that $(1+\varepsilon)^{-1}\| y \| \leqq \| Ty \| \leqq (1+\varepsilon) \| y \|$ for every $y \in l_n^\infty$ and such that the distance of x from $T l_n^\infty$ is $< \varepsilon$ for every $x \in A$. Here l_n^∞ denotes the space of all the n-tuples of real numbers $y = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ with $\| y \| = \max_i | \lambda_i |$. These spaces were called in [3] a^∞ spaces. Since l_n^∞ is a $\mathcal{P}_1$ space for every n it follows easily that an a^∞ is an $\mathcal{N}_\lambda$ space for every $\lambda > 1$. Here we show that the class of a^∞ spaces coincides with the class of the spaces which are $\mathcal{N}_\lambda$ for every $\lambda > 1$.

We consider only Banach spaces over the reals, but our result and its proof are valid also in the complex case. We state now our main result.

THEOREM 1. *Let X be a Banach space. Then the following three statements are equivalent.*

(i) *X^* is isometric to the space $L_1(\mu)$ for some measure μ.*

(ii) *X is an $\mathcal{N}_\lambda$ space for every $\lambda > 1$.*

(iii) *X is an a^∞ space.*

For spaces X whose unit cell has at least one extreme point Theorem 1 can be also easily deduced from the results of [1]. The proof of Theorem 1 presented

Received August 19, 1966.

* The research of the second named author has been sponsored by the Air Force Office of scientific Research under Grant AF EOAR 66-18 through the European Office of Aerospace Research (OAR) United States Air Force.

here is, however, shorter than the arguments given in [1] from which the special case of Theorem 1 follows.

A list of other properties equivalent to property (i) of Theorem 1 is given in [2, Theorem 6.1].

By combining the results of [3] with Theorem 1 we get immediately the following stronger version of Theorem 1 for separable spaces.

THEOREM 2. *Let X be a separable Banach space. Then the following two statements are equivalent.*

(i) *X^* is isometric to the space $L_1(\mu)$ for some measure μ.*

(ii) *X has a monotone basis $\{e_i\}_{i=1}^{\infty}$ such that for every n the subspace of X spanned by $\{e_i\}_{i=1}^{n}$ is isometric to l_n^{∞}.*

We pass to the proof of Theorem 1. As we have already remarked we need only to show that (i) $\rightarrow$ (iii). Let X satisfy (i) of Theorem 1, let A be a finite subset of X and let $0 < \varepsilon < 1$. In the definition of an a^{∞} space it is clearly enough to consider sets A with $\|x\| = 1$ for every $x \in A$ (otherwise replace $x \in A$ by $x/\|x\|$ and ε by $\varepsilon/\max_{x \in A}\|x\|$). So we assume that $\|x\| = 1$ for every $x \in A$ and let B be the subspace of X spanned by A. Let E_0 be the set of exposed points of the unit cell of B^*. Let $\tilde{E}_0$ be the set obtained from E_0 by identifying every f with $-f$, and let ϕ be the quotient map $\phi: E_0 \rightarrow \tilde{E}_0$. We metrize $\tilde{E}_0$ by putting $d(\phi f, \phi g) = \min(\|f - g\|, \|f + g\|)$. Since B is finite-dimensional the metric space $\tilde{E}_0$ is totally bounded. Hence, there is a finite number of subsets $\{\tilde{G}_i\}_{i=1}^{n}$ of $\tilde{E}_0$ such that $\tilde{G}_i \cap \tilde{G}_j = \emptyset$ for $i \neq j$, $\tilde{E}_0 = \bigcup_{i=1}^{n} \tilde{G}_i$ and $\tilde{G}_i$ has for every i a non empty interior and a diameter $< \varepsilon$. Since $\varepsilon < 1$ there is for every i a subset G_i of E_0 such that $\phi^{-1}\tilde{G}_i = G_i \cup -G_i$, $G_i \cap -G_i = \emptyset$ and $\|f - g\| < \varepsilon$ for every $f, g \in G_i$. For every i pick an $f_i \in G_i$ such that ϕf_i is an interior point of $\tilde{G}_i$ and let $x_i \in B$ be such that $f_i(x_i) = \|x_i\| = \|f_i\| = 1$ and $f(x_i) < 1$ for every $f \neq f_i$ in B^* with $\|f\| = 1$.

Let $E = \bigcup_{i=1}^{n} G_i$ and let $l^{\infty}(E)$ be the Banach space of all real-valued bounded functions on E with the sup norm. Let the operator $U: B \rightarrow l^{\infty}(E)$ be defined by $Ub(f) = f(b)$, $b \in B$, $f \in E$. Since the unit cell of B^* is the closed convex hull of $E \cup -E$ we get that U is an isometry. From our choice of the x_i and f_i it follows that there is a $\delta > 0$ such that $|f(x_i)| < 1 - \delta$ for every i and every $f \in E \sim G_i$. We assume as we may that $\delta < \min(2/3, 1 - \varepsilon)$.

Let $y_i \in l^{\infty}(E)$, $1 \leqq i \leqq n$, be defined by $y_i(f) = 1$ if $f \in G_i$ and $y_i(f) = 0$ if $f \in E \sim G_i$. By our choice of δ we get that $\|\delta^{-1}Ux_i - y_i\| \leqq \delta^{-1} - 1$. In fact, if $f \in E \sim G_i$ then

$$|\delta^{-1}Ux_i(f) - y_i(f)| = |\delta^{-1}f(x_i)| \leqq \delta^{-1} - 1,$$

while for $f \in G_i$ we get (since $\delta^{-1}f(x_i) \geqq (1 - \varepsilon)/\delta \geqq 1$)

$$|\delta^{-1}Ux_i(f) - y_i(f)| = |\delta^{-1}f(x_i) - 1| \leqq \delta^{-1} - 1.$$

Since X^* is an L_1 space there is (see e.g. [2, Theorem 6.1 (3)]) an operator T from $l^\infty(E)$ into X whose restriction to UB is equal to U^{-1} and with norm $\| T \| < (1 - \delta + \delta\varepsilon/2)/(1 - \delta)$. We have, in particular, that for every $1 \leqq i \leqq n$

$$\| \delta^{-1} x_i - Ty_i \| = \| \delta^{-1} TUx_i - Ty_i \| \leqq \| T \| \| \delta^{-1} Ux_i - y_i \| \leqq \delta^{-1} - 1 + \varepsilon/2,$$

and hence since $\| x_i \| = 1$ we get that $\| Ty_i \| \geqq 1 - \varepsilon/2$.

Let Y be the subspace of $l^\infty(E)$ spanned by $\{y_i\}_{i=1}^n$. Clearly, Y is isometric to l_n^∞. We claim that for every $y \in Y$

$$(1 - 2\varepsilon) \| y \| < \| Ty \| < (1 + \varepsilon) \| y \| .$$

That $\| T \| < 1 + \varepsilon$ follows from our choice of $\| T \|$ (observe that we assume that $\delta < 2/3$). Let now $y = \sum_{i=1}^n \lambda_i y_i \in Y$ with $\| y \| = 1$. Without loss of generality we may assume that $\lambda_1 = 1$. Let $z = y_1 - \sum_{i=2}^n \lambda_i y_i$.

Then $\| T(y + z) \| = 2 \| Ty_1 \| > 2 - \varepsilon$ and hence since $\| Tz \| < 1 + \varepsilon$ we get that $\| Ty \| > 1 - 2\varepsilon$.

In order to conclude the proof that X is an a^∞ space it is now enough to show that for every $x \in B$ with $\| x \| = 1$ there is a $y \in Y$ with $\| Ty - x \| < 2\varepsilon$. Take $y = \sum_{i=1}^n f_i(x) y_i \in Y$. Then $\| y - Ux \| \leqq \varepsilon$ (recall that the diameter of each G_i is $< \varepsilon$) and hence

$$\| Ty - x \| \leqq \| T \| \| y - Ux \| \leqq \varepsilon(1 + \varepsilon) < 2\varepsilon$$

and this concludes the proof.

References

1. A.J. Lazar, *Spaces of affine continuous functions on simplexes* (to appear).

2. J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48** (1964),

3. E. Michael and A. Pełczyński, *Separable Banach spaces which admit l_n^∞ approximations*, Israel J. Math. **4** (1966), 189–198.

The Hebrew University
of Jerusalem